

COHOMOLOGY AND REMOVABLE SUBSETS

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ABSTRACT. Let X be a (connected and reduced) complex space. A q -collar of X is a bounded domain whose boundary is a union of a strongly q -pseudoconvex, a strongly q -pseudoncave and two flat (i.e. locally zero sets of pluriharmonic functions) hypersurfaces. Finiteness and vanishing cohomology theorems obtained in [17], [18] for semi q -coronae are generalized in this context and lead to results on extension problem and removable sets for sections of coherent sheaves and analytic subsets.

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1. INTRODUCTION.

Let X be a (connected and reduced) complex space. We recall that X is said to be *strongly q -pseudoconvex* in the sense of Andreotti-Grauert [3] if there exist a compact subset $K \subset X$ and a smooth function $\varphi : X \rightarrow \mathbb{R}$, $\varphi \geq 0$, which is strongly q -plurisubharmonic on $X \setminus K$ and such that:

- a) $0 = \min_X \varphi < \min_K \varphi$;
- b) for every $c > \max_K \varphi$ the subset

$$B_c = \{x \in X : \varphi(x) < c\}$$

Date: February 4, 2008.

2000 Mathematics Subject Classification. Primary 32D15, 32D20 Secondary 32C25, 32F10.

Key words and phrases. Cohomology · pseudoconvexity · analytic extension · removable singularities.

is relatively compact in X .

If $K = \emptyset$, X is said to be *q-complete*. We remark that, for a space, being 1-complete is equivalent to being Stein.

Replacing the condition b) by

b') for every $0 < a < \min_K \varphi$ and $c > \max_K \varphi$ the subset

$$B_{a,c} = \{x \in X : a < \varphi(x) < c\}$$

is relatively compact in X ,

we obtain the notion of *q-corona* (see [3], [4]). A *q-corona* is said to be *complete* whenever $K = \emptyset$.

The extension problem for analytic objects (basically, sections of coherent sheaves, cohomology classes, analytic subsets) defined on *q-coronae* was studied by many authors (see e.g. [3], [11], [19], [20], [21]).

In [17], [18] we dealt with the larger class of the *semi q-coronae* which are defined as follows. Consider a strongly *q-pseudoconvex* space (or, more generally, a *q-corona*) X , and a smooth function $\varphi : X \rightarrow \mathbb{R}$ displaying the *q-pseudoconvexity* of X . Let $B_{a,c} \subset X$ and let $h : X \rightarrow \mathbb{R}$ be a pluriharmonic function such that $K \cap \{h = 0\} = \emptyset$. A connected component of $B_{a,c} \setminus \{h = 0\}$ is, by definition, a *semi q-corona*. If X is a complex manifold the zero set $\{h = 0\}$ can be replaced by a Levi flat hypersurface. For singular spaces, by definition *Levi flatness* means locally zero set of a pluriharmonic function.

Finiteness and vanishing cohomology theorems proved there lead to results of this type: depending on q , analytic objects given near the convex part of the boundary of a *semi q-corona* fill in the hole.

In this paper we consider a more general situation. Let X be a strongly *q-pseudoconvex* space, $C = B_{a,c} = B_c \setminus \overline{B}_a$ a *q-corona*. Let Σ_1, Σ_2 two Levi-flat hypersurfaces in a neighbourhood of \overline{B}_c such that

$$B_c \cap \Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap K = \Sigma_2 \cap K = \emptyset,$$

and $\Sigma_1 \cap B_c \neq \emptyset$, $\Sigma_2 \cap B_c \neq \emptyset$ are nonempty connected subsets. We also assume that $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1, h_2 are pluriharmonic on neighbourhoods W_1, W_2 of $\Sigma_1 \cap B_c, \Sigma_2 \cap B_c$ respectively. Let Q be the open subset of B_c bounded by $\Sigma_1 \cap B_c, \Sigma_2 \cap B_c$ and a part of ∂B_c . We assume that Q is connected and that $B_c \setminus \overline{Q}$ has two connected components, B_+ and B_- , and define $C_0 = Q \cap C$, $C_+ = B_+ \cap C$, $C_- = B_- \cap C$. The domain C_0 is called a *q-collar* (see fig. 1). A *q-collar* is said to be *complete* if $K = \emptyset$. Note that C_+ and C_- are semi *q-coronae*.

Observe that *q-collar* is a difference of two strongly pseudoconvex spaces. Indeed, consider $1/(c - \varphi)$ which is a strongly *q-plurisubharmonic* exhaustion function for B_c . We may suppose that the functions h_1, h_2 are smooth on all of X . Moreover, $\psi_1 = -\log h_1^2$, $\psi_2 = -\log h_2^2$ are plurisubharmonic in $W_1 \setminus \{h_1 = 0\}, W_2 \setminus \{h_2 = 0\}$ respectively. Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing convex function such that $\chi \circ (1/(c - \varphi)) > \psi$ on a neighbourhood of

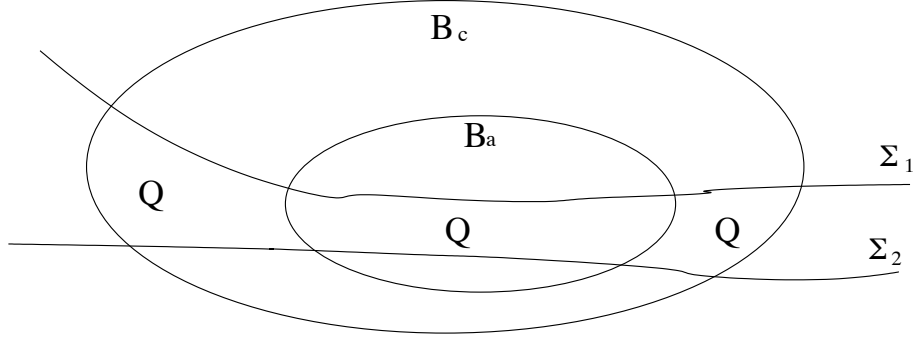


FIGURE 1. A q -collar $C_0 = Q \cap (B_c \setminus \overline{B_a})$. In spite of the figure, C_0 is connected.

$B_c \setminus W_1$. The function

$$\Phi_1 = \sup (\chi \circ (1/c - \varphi), \psi) + \frac{1}{c - \varphi}$$

is an exhaustion function for $B_c \setminus \{h_1 = 0\}$ which is strongly q -plurisubharmonic in $B_c \setminus (\{h_1 = 0\} \cup K)$. In a similar way we construct an exhaustion function Φ_2 for $B_c \setminus \{h_2 = 0\}$ which is strongly q -plurisubharmonic in $B_c \setminus (\{h_2 = 0\} \cup K)$. Then the function $\Phi = \sup (\Phi_1, \Phi_2)|_Q$ is an exhaustion function for Q which is strongly q -plurisubharmonic in $Q \setminus K$. In order to get the conclusion it is sufficient to apply the same argument starting from B_a .

If $\mathcal{F} \in \text{Coh}(B_c)$, we define

$$p(\mathcal{F}) = \inf_{x \in B_c} \text{depth}(\mathcal{F}_x),$$

the depth of \mathcal{F} on B_c . If $\mathcal{F} = \mathcal{O}$, the structure sheaf of X , we set $p(B_c) = p(\mathcal{O})$.

The results on the cohomology of q -collars, generalizing the ones proved in [17], [18], are established in the first part of the paper (see Section 2). They are applied in Section 3 to study removability. Removability for functions was extensively studied by many authors (see e.g. [22], [16], [13], [8]). We are dealing with removability for sections of coherent sheaves and analytic sets. The main results are contained in Theorems 8, 9, 10, 11.

2. SOME COHOMOLOGY

This section is dealing with cohomology of q -collars and some application to extension of sections of coherent sheaves.

2.1. Closed q -collars. Let C_0 be a q -collar in a strongly q -pseudoconvex space X .

Theorem 1. *Let $\mathcal{F} \in \text{Coh}(B_c)$. Then, for $q-1 \leq r \leq p(\mathcal{F})-q-2$, the homomorphism*

$$H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}_0, \mathcal{F})$$

(all closures are taken in B_c), defined by $(\xi \oplus \eta) \mapsto \xi|_{\overline{C}_0} - \eta|_{\overline{C}_0}$, has finite codimension.

If $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1 and h_2 are pluriharmonic functions near $\Sigma_1 \cap \overline{B}_c$ and $\Sigma_2 \cap \overline{B}_c$, respectively, then

$$\dim_{\mathbb{C}} H^r(\overline{C}_0, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F})-q-2$.

Proof. Consider the Mayer-Vietoris sequence applied to the closed sets \overline{Q} and \overline{C}

$$(1) \quad \begin{aligned} \cdots &\rightarrow H^r(\overline{Q} \cup \overline{C}, \mathcal{F}) \rightarrow H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \xrightarrow{\delta} \\ &\xrightarrow{\delta} H^r(\overline{C}_0, \mathcal{F}) \rightarrow H^{r+1}(\overline{Q} \cup \overline{C}, \mathcal{F}) \rightarrow \cdots \end{aligned}$$

$\delta(\xi \oplus \eta) = \xi|_{\overline{C}_0} - \eta|_{\overline{C}_0}$. We have

$$\overline{Q} \cup \overline{C} = B_c \setminus U, =$$

where $U = B_a \setminus (B_a \cap \overline{Q})$. Thus U is q -complete and consequently the groups of compact support cohomology $H_c^r(U, \mathcal{F})$ are zero for $q \leq r \leq p(\mathcal{F})-q$ [3].

From the exact sequence of compact support cohomology

$$(2) \quad \begin{aligned} \cdots &\rightarrow H_c^r(U, \mathcal{F}) \rightarrow H^r(B_c, \mathcal{F}) \rightarrow \\ &\rightarrow H^r(B_c \setminus U, \mathcal{F}) \rightarrow H_c^{r+1}(U, \mathcal{F}) \rightarrow \cdots \end{aligned}$$

it follows that

$$(3) \quad H^r(B_c, \mathcal{F}) \xrightarrow{\sim} H^r(B_c \setminus U, \mathcal{F}),$$

for $q \leq r \leq p(\mathcal{F})-q-1$.

Since B_c is q -pseudoconvex,

$$\dim_{\mathbb{C}} H^r(B_c, \mathcal{F}) < \infty$$

for $q \leq r$ [3], and so

$$\dim_{\mathbb{C}} H^r(B_c \setminus U, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F})-q-1$.

From (1) we see that

$$\dim_{\mathbb{C}} H^r(B_c \setminus U, \mathcal{F}) = \dim_{\mathbb{C}} H^r(\overline{Q} \cup \overline{C}, \mathcal{F})$$

is greater than or equal to the codimension of the homomorphism δ . This proves that the image of the homomorphism

$$H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}_0, \mathcal{F})$$

(all closures are taken in B_c), defined by $(\xi \oplus \eta) \mapsto \xi|_{\overline{C}_0} - \eta|_{\overline{C}_0}$ has finite codimension provided that $q - 1 \leq r \leq p(\mathcal{F}) - q - 2$, proving the first assertion of the theorem.

If $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ are like in the second part of the statement, then, since $K \cap (\Sigma_1 \cup \Sigma_2) = \emptyset$, \overline{Q} has a fundamental system of neighborhoods which are q -pseudoconvex spaces, thus, by virtue of [3, Théorème 11] we have

$$\dim_{\mathbb{C}} H^r(\overline{Q}, \mathcal{F}) < \infty$$

for $r \geq q$. On the other hand, \overline{C} is a q -corona, so

$$\dim_{\mathbb{C}} H^r(\overline{C}, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$ in view of [4, Theorem 3].

Summarizing, for $q \leq r \leq p(\mathcal{F}) - q - 1$ the vector space $H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F})$ has finite dimension and for $q - 1 \leq r \leq p(\mathcal{F}) - q - 2$ its image in $H^r(\overline{C}_0, \mathcal{F})$ has finite codimension. Thus, for $q \leq r \leq p(\mathcal{F}) - q - 2$, $H^r(\overline{C}_0, \mathcal{F})$ has finite dimension. \square

Theorem 2. Assume that $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1 and h_2 are pluriharmonic functions near $\Sigma_1 \cap \overline{B}_c$ and $\Sigma_2 \cap \overline{B}_c$, respectively, and $\overline{Q} \cap K = \emptyset$. Then

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_0, \mathcal{F})$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$ and the homomorphism

$$(4) \quad H^{q-1}(\overline{Q}, \mathcal{F}) \oplus H^{q-1}(\overline{C}, \mathcal{F}) \longrightarrow H^{q-1}(\overline{C}_0, \mathcal{F})$$

is surjective for $p(\mathcal{F}) \geq 2q + 1$.

If \overline{B}_+ is a 1-complete space and $p(\mathcal{F}) \geq 3$, the homomorphism

$$H^0(\overline{Q}, \mathcal{F}) \longrightarrow H^0(\overline{C}_0, \mathcal{F})$$

is surjective.

Proof. By hypothesis \overline{Q} has a fundamental system of neighborhoods which are q -complete spaces, so $H^r(\overline{Q}, \mathcal{F}) = \{0\}$ for $q \leq r$ [3, Théorème 5]. From (3) it follows that $H^r(\overline{Q} \cup \overline{C}_0, \mathcal{F}) = \{0\}$ for $q \leq r \leq p(\mathcal{F}) - q - 1$. Thus, from the Mayer-Vietoris sequence (1) we derive the isomorphism

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_+, \mathcal{F})$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$ and that the homomorphism (4) is surjective if $p(\mathcal{F}) \geq 2q + 1$.

In particular, if $q = 1$ and $p(\mathcal{F}) \geq 3$ the homomorphism

$$H^0(\overline{Q}, \mathcal{F}) \oplus H^0(\overline{C}, \mathcal{F}) \longrightarrow H^0(\overline{C}_0, \mathcal{F})$$

is surjective, i.e. every section $\sigma \in H^0(\overline{C}_0, \mathcal{F})$ is a difference $\sigma_1 - \sigma_2$ of two sections $\sigma_1 \in H^0(\overline{Q}, \mathcal{F})$, $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$. Since B_a is Stein, the cohomology group with compact supports $H_c^1(B_a, \mathcal{F})$ is zero, and so the Mayer-Vietoris compact support cohomology sequence implies that the restriction

homomorphism

$$H^0(\overline{B}_c, \mathcal{F}) \longrightarrow H^0(\overline{B}_c \setminus B_a, \mathcal{F}) = H^0(\overline{C}, \mathcal{F})$$

is surjective, hence $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$ is restriction of $\tilde{\sigma}_2 \in H^0(B_c, \mathcal{F})$. So σ is restriction to \overline{C}_+ of $(\sigma_1 - \tilde{\sigma}_2|_{\overline{B}_+}) \in H^0(\overline{Q}, \mathcal{F})$, and the restriction homomorphism is surjective. \square

Corollary 3. *Let $q = 1$ and $p(B_c) \geq 3$. Then every holomorphic function on \overline{C}_0 extends holomorphically on \overline{Q} .*

2.2. Open q -collars. Keeping the same notations as above consider an open q -collar C_0 . For the sake of simplicity we assume that B_c is q -complete. We also assume that $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1 and h_2 are pluriharmonic functions on open neighbourhoods U_1 and U_2 of $\Sigma_1 \cap \overline{B}_c$ and $\Sigma_1 \cap \overline{B}_c$, respectively.

Theorem 4. *Let B_c be 1-complete and \mathcal{F} a coherent sheaf on B_c with $p(\mathcal{F}) \geq 3$. Then the homomorphism*

$$H^0(Q, \mathcal{F}) \longrightarrow H^0(C_0, \mathcal{F})$$

is surjective.

Proof. Let $s \in H^0(C_0, \mathcal{F})$. Fix a couple of positive numbers $\varepsilon = (\varepsilon_1, \varepsilon_2)$ small enough such that $\Sigma_{i, \varepsilon_i}$ defined by $\Sigma_{i, \varepsilon_i} = \{h_i = \varepsilon_i\}$ are connected hypersurfaces, $\Sigma_{i, \varepsilon_i} \cap \overline{B}_c \cap \overline{Q} \neq \emptyset$ and $\Sigma_{i, \varepsilon_i} \cap \overline{B}_c \subset U_i$, for $i = 1, 2$.

Consider the open subset Q_ε of Q bounded by the hypersurfaces $\Sigma_{i, \varepsilon_i} \cap \overline{B}_c$, and by a part of bB_c , and set $C_{0, \varepsilon} = Q_\varepsilon \cap C_0$. In view of Theorem 2 there exists a section $\tilde{s}_\varepsilon \in H^0(\overline{Q}_\varepsilon, \mathcal{F})$ which extends $s|_{C_{0, \varepsilon}}$. Now observe that the connected component W of $B_c \setminus \Sigma_1$ containing Σ_2 is Stein. So there exists a strongly pseudoconvex domain $\Omega \Subset W$ such that the domain D_ε bounded by $\Sigma_{2, \varepsilon_2} \cap \overline{B}_c$, $\Sigma_2 \cap \overline{B}_c$ and by a part of bB_c is relatively compact in Ω . By Theorem 5 of [17] the section \tilde{s}_ε extends on $\Omega \cap Q$. Thus s extends on Q_ε . In order to conclude the proof we argue as before with respect to the hypersurfaces $\Sigma_{1, \varepsilon_1}$ and Σ_1 . \square

In particular, we get the extension of holomorphic functions:

Corollary 5. *If B_c is a 1-complete space and $p(B_c) \geq 3$, every holomorphic function on C_0 can be holomorphically extended on Q .*

Corollary 6. *Let X be a Stein space. Let $\Sigma_1 = \{h_1 = 0\} \subset X$, and $\Sigma_2 = \{h_2 = 0\} \subset X$ be the zero set of two pluriharmonic functions, and S be a real hypersurface of X with boundary, such that $S \cap \Sigma_1 = bS = bA_1$, $S \cap \Sigma_2 = bS = bA_2$ where A_1 is an open set in Σ_1 and A_2 is an open set in Σ_2 . Let $D \subset X$ be the relatively compact domain bounded by $S \cup A_1 \cup A_2$ and \mathcal{F} be a coherent sheaf with $\text{depth}(\mathcal{F}) \geq 3$. All sections of \mathcal{F} on S extend to D .*

2.3. Finiteness of cohomology. Results on the cohomology of q -collars obtained in the preceding section concern coherent sheaves defined in larger domains. For the applications that we have in mind it is needed to study cohomology of coherent sheaves which are defined just on collars. This can be done by the same methods used in [18] for semi q -coronae. We briefly sketch the main points of proofs given there focusing on the case $q = 1$. The extension for an arbitrary q demands only technical adjustments. Keeping the same notations as in Section 1 let

$$C_0 = Q \cap (B_c \setminus \bar{B}_a) = Q \cap B_{a,c} = Q \cap \{x \in X : a < \varphi(x) < c\}$$

be an open 1-collar of a Stein space X (see fig. 1, page 3). Q is the subdomain of B_c bounded by the two Levi flat hypersurfaces $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$. Σ_1 and Σ_2 are defined on a neighbourhood of \bar{B}_c where h_1 and h_2 are pluriharmonic functions near Σ_1 and Σ_2 respectively. Thus Q is a Stein domain. By Σ_1^0, Σ_2^0 we denote the Levi flat parts of bC_0 and by F_1^0, F_2^0 the 1-pseudoconvex and the 1-pseudoconcave part respectively. Since Q is Stein, there exist two families of 1-pseudoconvex hypersurfaces $\{\Sigma_1^\varepsilon\}, \{\Sigma_2^\varepsilon\}$, $\varepsilon \searrow 0$, in a neighbourhood of \bar{Q} , with the following properties

- 1) $\Sigma_1^\varepsilon, \Sigma_2^\varepsilon$ bound a strip $Q_\varepsilon \subset Q$ and $\Sigma_1^\varepsilon \rightarrow \Sigma_1, \Sigma_2^\varepsilon \rightarrow \Sigma_2$ as $\varepsilon \searrow 0$;
- 2) defining $C_0^\varepsilon = Q_\varepsilon \cap B_{a+\varepsilon, c-\varepsilon}$ we obtain an exhaustion $\{C_0^\varepsilon\}$ of the collar C_0 .

Bump lemma and approximation theorem hold for the closed subsets \bar{C}_0^ε with the same proof as in [18, Lemma 3.3, 3.9] and this enables us to the following results. Assume that $\text{depth } \mathcal{F}_z \geq 3$ for z near to the pseudoconcave part of the boundary of C_0 ; then

- 3) there exists ε_0 sufficiently small such that if $\varepsilon < \varepsilon_0$ the cohomology spaces $H^1(\bar{C}_\varepsilon^+, \mathcal{F})$ are finite dimensional;
- 4) if $\varepsilon < \varepsilon_0$ there exists $\varepsilon_1 < \varepsilon$ such that

$$H^1(C_{\varepsilon'}^+, \mathcal{F}) \simeq H^1(\bar{C}_\varepsilon^+, \mathcal{F})$$

for every $\varepsilon' \in]\varepsilon_1, \varepsilon[$.

3), 4) have an important consequence, namely that for \mathcal{F} Theorem A of Oka-Cartan-Serre holds in the following form (see [18, Corollary 4.2]):

- 5) if $\varepsilon, \varepsilon'$ are as in 4), for every compact subset K of $C_{\varepsilon'}^+ \setminus \{\varphi > c - \varepsilon\}$ there exist sections $s_1, \dots, s_k \in H^0(C_{\varepsilon'}^+, \mathcal{F})$ which generate \mathcal{F}_z for every $z \in K$.

As an application we get the following extension theorem for analytic subsets

Theorem 7. *Let X be a Stein space, $C_0 = Q \cap (B_c \setminus \bar{B}_a) \subset X$ be a complete 1-collar and Y be a closed analytic subset of C_0 such that $\text{depth}(\mathcal{O}_{Y,z}) \geq 3$ for z near $\{\varphi = a\}$. Then Y extends to a closed analytic subset on Q .*

Proof. Taking into account 5) the proof runs as in [18, Theorem 4.3 and Corollary 4.4]. \square

3. REMOVABLE SETS

The notion of removable sets was originally given with respect to holomorphic function and the removability problem was extensively studied (see e.g. [22], [16], [13], [8]). Here we want to study the same problem with respect to larger classes of analytic objects, namely the classes of sections of coherent sheaves, of cohomology classes and of analytic sets.

Let X be a complex space, D be a bounded domain. Let \mathcal{F} be a coherent sheaf on a neighbourhood of \bar{D} . A subset L of the boundary $\text{b}D$ of D is said to be *removable* for (the sections of) \mathcal{F} or for the cohomology classes with value in \mathcal{F} , of a certain degree r , if every section $s \in \Gamma(\text{b}D \setminus L, \mathcal{F})$ or cohomology class $\omega \in H^r(\text{b}D \setminus L, \mathcal{F})$ extends by $\tilde{s} \in \Gamma(\bar{D} \setminus L, \mathcal{F})$ or by $\tilde{\omega} \in H^r(\bar{D} \setminus L, \mathcal{F})$ respectively.

Similarly, the subset L is said to be *removable* for the (respectively, a given) class of analytic subsets if every analytic subset (of a given class of analytic subsets) defined on a neighbourhood of $\text{b}D \setminus L$ extends by an analytic subset of $\bar{D} \setminus L$.

3.1. Coherent sheaves. Given a coherent sheaf \mathcal{F} on a complex space X let us denote $\text{Tor}(\mathcal{F})$ the torsion of \mathcal{F} ; $\text{Tor}(\mathcal{F})$ is the coherent subsheaf of \mathcal{F} whose stalk at a point $x \in X$ is

$$\text{Tor}(\mathcal{F})_x = \{s_x \in \mathcal{F}_x : \lambda_x s_x = 0 \text{ for some } \lambda \in \mathcal{O}_x, \lambda \neq 0\}.$$

It can be proved (see [2]) that the topology of \mathcal{F} is Hausdorff if and only if \mathcal{F} has no torsion, i.e. $\text{Tor}(\mathcal{F}) = \{0\}$. We denote $T(\mathcal{F})$ the analytic subset $\text{supp Tor}(\mathcal{F})$.

Given a bounded domain $D \subset X$ let $\mathcal{A}(D)$ be the algebra $C^0(\bar{D}) \cap \mathcal{O}(D)$ and for every compact $L \subset \bar{D}$ let

$$\hat{L} = \left\{ z \in \bar{D} : |f(z)| \leq \max_L |f|, \forall f \in \mathcal{A}(D) \right\}$$

be the $\mathcal{A}(D)$ -envelope of L . We want to prove the following

Theorem 8. *Let X be an n -dimensional manifold, D a bounded pseudoconvex domain in X with a connected smooth boundary and L a compact subset of $\text{b}D$ such that $\text{b}D \setminus L$ is a connected, nonempty strongly Levi convex hypersurface. Let \mathcal{F} be a coherent sheaf on X . Assume that:*

- 1) $D \setminus \hat{L}$ is connected;
- 2) $\text{depth}(\mathcal{F}_x) \geq 3$ for every $x \in \bar{D}$;
- 3) $\dim_{\mathbb{C}} T(\mathcal{F}) \cap \bar{D} \leq n - 2$

Let U be an open neighborhood of $\bar{D} \setminus L$ or $X \setminus (D \cup L)$. Then every section of \mathcal{F} on $U \setminus \bar{D}$ or $U \setminus (X \setminus D)$ uniquely extends to a section on $U \setminus \hat{L}$ or $D \setminus \hat{L}$. In particular, if $\hat{L} = L$ then L is removable for \mathcal{F} .

Proof. The uniqueness is a consequence of the *Kontinuitätsatz* and of hypothesis 1) and 2). Indeed let s_1, s_2 be sections of \mathcal{F} on D such that $s_1 \equiv s_2$

near $bD \setminus L$. In view of the hypothesis 2), the support of $s_1 - s_2$ is an analytic subset A of $D \setminus \widehat{L}$ with no 0-dimensional irreducible component (see [5, Théorème 3.6 (a), p. 46]). Let A_1 be an irreducible component of A . Since $bD \setminus L$ is strongly Levi convex, in view of the *Kontinuitätsatz* A_1 cannot touch $bD \setminus L$ so $\overline{A_1} \cap bD \equiv \overline{A_1} \cap (bD \setminus L)$. Let $x \in A_1$. In view of the hypothesis 1) there exists $f \in \mathcal{A}(D)$ such that $\max_L |f| < |f(x)|$. Consider an exhaustion $W_1 \Subset W_2 \Subset \dots$ by relatively open subsets of A_1 , $x \in W_1$. By virtue of the maximum principle, for every k there exists a point $x_k \in bW_k$ such that $|f(x)| < |f(x_k)|$. Then (passing if necessary to a subsequence) we have $x_k \rightarrow y \in \widehat{L}$ as $k \rightarrow +\infty$ and consequently $|f(x)| \leq |f(y)| \leq \max_L |f|$, a contradiction.

We need now to show the existence of the extension. In order to prove the extension we consider just the case that U is an open neighborhood of $\overline{D} \setminus L$ or $X \setminus (D \cup L)$ and $\sigma \in \mathcal{F}(U \setminus \overline{D})$, the proof in the other one being similar. In view of the hypothesis 1), given a point $x \in D \setminus \widehat{L}$ there exists $f \in \mathcal{A}(D)$, $f = u + iv$, u, v real-valued functions, such that $f(x) = u(x) = 1$, $\max_L |f| < 1$; in particular $\max_L |u| < 1$. Then, if $\varepsilon > 0$ is sufficiently small and $C = \{u \leq 1 - \varepsilon\}$, we have $C \cap L = \emptyset$. Let V be an open neighborhood of L such that $C \cap \overline{V} = \emptyset$. Since $bD \setminus L$ is strongly pseudoconvex, there exists a pseudoconvex domain D_1 with a smooth boundary satisfying the following properties:

- i) $D \subset D_1, \overline{D_1} \setminus D \subset U$;
- ii) $bD_1 \cap bD \subset V \cap bD$;
- iii) bD_1 is strongly pseudoconvex at the points of $bD_1 \setminus bD_1 \cap bD$.

Since D_1 is Stein there exists a strongly pseudoconvex $D_2 \Subset D_1$ which contains the compact subset $\overline{D} \setminus V \cap D$ and such that $b(D_2 \cap D) \setminus bD \Subset W$ (see fig. 2).

The boundary of $D_3 = D_2 \cap D$ is piecewise smooth but we may regularize it along $bD_2 \cap bD$, thus we may assume that D_3 is a smooth strongly pseudoconvex domains $D_3 = \{\rho < 0\}$, where ρ is a strongly plurisubharmonic function on a neighbourhood of $\overline{D_3}$ and $d\rho(z) \neq 0$ along bD_3 . By the approximation theorem of Kerzman (see [14]) there exists an open neighbourhood W of $\overline{D_3}$ such that $\mathcal{O}(W)$ is a dense subalgebra of $\mathcal{A}(D_3)$. It follows that we may assume that:

- a) $\sigma \in \mathcal{F}(bD_3 \cap \{u > 1 - \varepsilon\})$, where u is pluriharmonic near $\overline{D_3}$;
- b) $\{u = 1 - \varepsilon\}$ is smooth and intersects bD_3 transversally;
- c) $\{u > 1 - \varepsilon\} \cap D_3$ has a finite number of connected components $D^{(1)}, \dots, D^{(k)}$, which are Stein domains whose boundaries consist of a part of bD_3 and of closed subsets contained in $\{u = 1 - \varepsilon\}$.

Moreover, we may suppose that $D^{(i+1)}$ and $D^{(i)}$, $1 \leq i \leq k-1$, are consecutive (i.e. there is a path $\gamma \subset D_3$ joining two points $y' \in D^{(i)}$, $y'' \in D^{(i+1)}$ which does not meet any other connected component $D^{(j)}$) and $x \in D^{(1)}$. We denote by Σ_i, Σ_{i+1} , $1 \leq i \leq k$, the flat parts of $D^{(i)}$; in particular $\Sigma_{k+1} = \emptyset$.

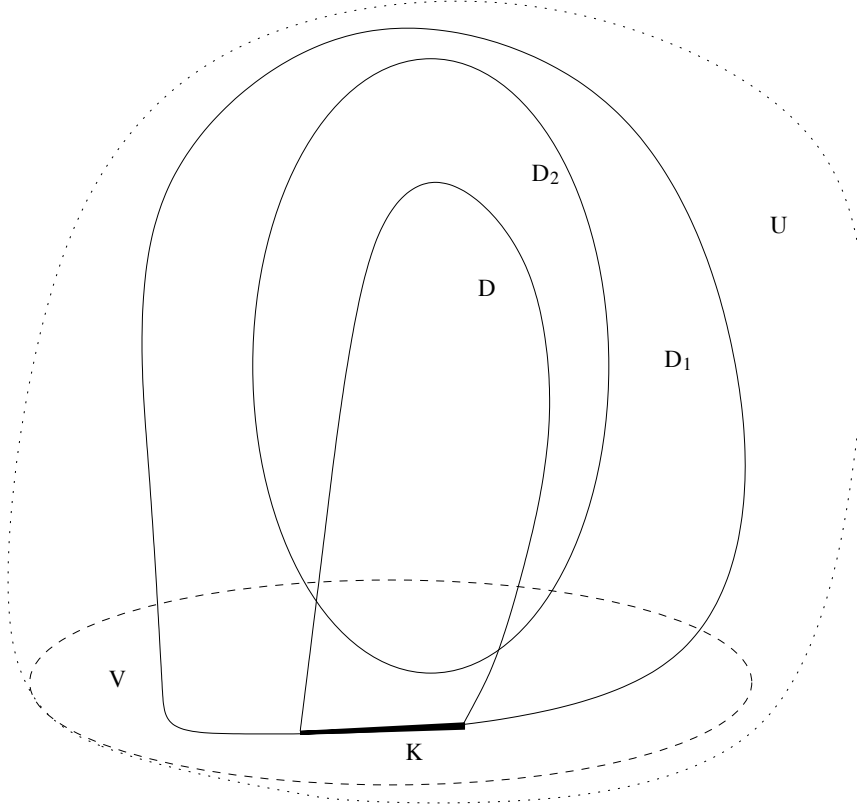


FIGURE 2. Construction of the three domains D_1 , D_2 and $D_3 = D_2 \cap D$.

Let us start by $D^{(k)}$. In view of the extension theorem proved in [17], there exists a unique section $\sigma_k \in \mathcal{F}(D^{(k)})$ which extends σ . Now consider a positive $\varepsilon' < \varepsilon$ such that the hypersurface $\{u = 1 - \varepsilon'\}$ is smooth, intersects bD_3 transversally and $D^{(k-1)}$ contains only one connected component Σ'_k of $\{u = 1 - \varepsilon'\}$. Since $\Omega = D^{(k-1)} \cup D^{(k)} \cup \Sigma_k$ is a Stein domain, there exists a strongly pseudoconvex domain $\Omega' \Subset \Omega$ with the following properties:

- d) Ω' contains the closed domain bounded by $\Sigma_k, \Sigma'_{k-1}, bD, b\Omega'$ intersects Σ'_k transversally;
- e) no connected component of $\{u = 1 - \varepsilon'\}, \Sigma'_{k-1}$ excepted, intersects Ω' .

Thanks again to the quoted extension theorem applied to the subdomain Ω' of Ω bounded by $b\Omega$ and intersecting Σ_k , we extend σ_k by σ'_k to Ω' . Arguing as above with respect to the domain bounded by $\Sigma'_{k-1}, \Sigma_{k-1}, bD$, we extend σ'_k to $D^{(k-1)}$ and so on.

In order to finish the proof we have to show that if $\tilde{\sigma}, \tilde{\sigma}'$ are two such extensions, defined on \tilde{U} and \tilde{U}' respectively, then $\tilde{\tau} = \tilde{\sigma}'$ on $\tilde{U} \cap \tilde{U}'$. This is trivially true if \mathcal{F} is locally isomorphic to a subsheaf of \mathcal{O}^N , in particular if \mathcal{F} is locally free.

In our situation consider the difference $\tilde{\tau} = \tilde{\sigma} - \tilde{\sigma}'$ on $\tilde{U} \cap \tilde{U}'$. Since \mathcal{F} is Hausdorff on $D \setminus T$, $T = T(\mathcal{F})$, $\text{supp } \tilde{\tau} \subset T$. Let $x \in \text{supp } \tilde{\tau}$. If $B \subset \tilde{U} \cap \tilde{U}'$ is a sufficiently small Stein neighbourhood of x we have the exact sequences

$$\mathcal{O}^p \xrightarrow{\psi} \mathcal{O}^q \xrightarrow{\varphi} \mathcal{F} \longrightarrow 0.$$

$$H^0(B, \mathcal{O}^p) \xrightarrow{\psi} H^0(B, \mathcal{O}^q) \xrightarrow{\varphi} H^0(B, \mathcal{F}) \longrightarrow 0.$$

where ψ, φ are defined by matrices $(\psi_{ij}), (\varphi_{rs})$ of holomorphic functions on B . Then $\tilde{\tau} = \varphi(s)$, $s = (s_1, \dots, s_q) \in H^0(B, \mathcal{O}^q)$ and $\varphi(s_y) = 0$ for every $y \in B \setminus T$; consequently

$$s|_{B \setminus T} \in H^0(B \setminus T, \text{Ker } \varphi) = H^0(B \setminus T, \text{Im } \psi).$$

It follows that there exist holomorphic functions g_1, \dots, g_p on $B \setminus T$ such that

$$s_1|_{B \setminus T} = \sum_{j=1}^p \psi_{1j} g_j, \dots, s_q|_{B \setminus T} = \sum_{j=1}^p \psi_{qj} g_j.$$

Since $\dim_{\mathbb{C}} T \leq n - 2$, the functions g_1, \dots, g_p can be holomorphically extended through T by $\tilde{g}_1, \dots, \tilde{g}_p$. This implies that $s \in H^0(B, \text{Im } \psi)$, so $s = \psi(\tilde{g})$, $g = (g_1, \dots, g_p)$, and consequently $\tilde{\sigma} = (\varphi \circ \psi)(\tilde{g}) = 0$.

The proof when U is a neighbourhood of $X \setminus (D \cup L)$ is similar starting by a pseudoconvex domain D_1 with a smooth boundary satisfying the following properties:

- i) $D_1 \subset D$, $D \setminus \overline{D}_1 \subset U$;
- ii) $bD_1 \cap bD \subset V \cap bD$;
- iii) bD_1 is strongly pseudoconvex at the points of $bD_1 \setminus bD_1 \cap bD$.

□

Remark 3.1. In view of a theorem by Alexander [1], condition 1) of Theorem 8 is certainly satisfied if $\hat{L} \cap bD = L$. Indeed, the connected components A_i of $D \setminus \hat{L}$ and B_i of $bD \setminus (\hat{L} \cap bD)$ are in a 1 – 1 correspondence given by

$$A_i \leftrightarrow B_i \iff bA_i \cap bD = B_i.$$

Since $L = \hat{L} \cap bD$, and $bD \setminus L$ is connected, also $D \setminus \hat{L}$ is connected.

Condition 1) of Theorem 8 can be dropped also if L is a Stein compact.

Theorem 9. *Let X be a locally irreducible Stein space, D be a bounded domain in X with a connected smooth boundary and $L \subset bD$ be a Stein compact such that $bD \setminus L$ is connected. Let \mathcal{F} be a coherent sheaf on X . Assume that:*

- 1) $\text{depth}(\mathcal{F}_x) \geq 3$ for every $x \in X$;
- 2) $\dim_{\mathbb{C}} T(\mathcal{F}) \leq n - 2$.

Let U be an open neighborhood of $\overline{D} \setminus L$. Then every section of \mathcal{F} on $U \setminus \overline{D}$ uniquely extends to a section on $U \setminus L$.

Proof. Let

$$p(\mathcal{F}) = \inf_{x \in X} \text{depth}(\mathcal{F}_x)$$

and $\{U_\alpha\}$ be a fundamental system of Stein neighbourhoods of L . Then for the compact support cohomology groups we have

$$H_c^j(U_\alpha, \mathcal{F}) = 0,$$

for $j \leq p(\mathcal{F}) - 1$ and every α . Moreover, if $H_L^j(X, \mathcal{F})$ denotes the j^{th} local cohomology group with support in L we have the isomorphism

$$H_L^j(X, \mathcal{F}) = \varinjlim_{U_\alpha} H_c^j(U_\alpha, \mathcal{F})$$

(see [5]) hence

$$H_L^j(X, \mathcal{F}) = \{0\}$$

for $j \leq p(\mathcal{F}) - 1$.

From the local cohomology exact sequence

$$\cdots \rightarrow H^j(X, \mathcal{F}) \rightarrow H^j(X \setminus L, \mathcal{F}) \rightarrow H_L^{j+1}(X, \mathcal{F}) \rightarrow \cdots,$$

in view of the fact X is a Stein space, we then obtain

$$H^j(X \setminus L, \mathcal{F}) = \{0\}$$

for $1 \leq j \leq p(\mathcal{F}) - 2$. In particular, since $p(\mathcal{F}) \geq 3$, we have

$$H^1(X \setminus L, \mathcal{F}) = \{0\}.$$

Let $s \in H^0(bD \setminus L, \mathcal{F})$. Applying the Mayer-Vietoris sequence to the following closed partition of $X \setminus L$

$$X \setminus L = (\overline{D} \setminus L) \cup [X \setminus (D \cup L)]$$

we get the exact sequence

$$H^0(\overline{D} \setminus L, \mathcal{F}) \oplus H^0(X \setminus (D \cup L), \mathcal{F}) \rightarrow H^0(bD \setminus L, \mathcal{F}) \rightarrow H^1(X \setminus L, \mathcal{F}).$$

Since $H^1(X \setminus L, \mathcal{F}) = \{0\}$ the first homomorphism is onto, so the section s is a difference $s = s_1 - s_2$ of two sections

$$s_1 \in H^0(\overline{D} \setminus L, \mathcal{F}), \quad s_2 \in H^0(X \setminus (D \cup L), \mathcal{F}).$$

Hence, in order to end our proof, we have to extend the section s_2 . Consider an open Stein neighbourhood U of L . Since, by hypothesis, $p(\mathcal{F}) \geq 3$, we have $H_c^1(U, \mathcal{F}) = \{0\}$ and consequently, again from the cohomology exact sequence

$$H^0(X, \mathcal{F}) \rightarrow H^0(X \setminus U, \mathcal{F}) \rightarrow H_c^1(U, \mathcal{F}) \rightarrow \cdots,$$

we deduce that the homomorphism

$$H^0(X, \mathcal{F}) \rightarrow H^0(X \setminus \overline{U}, \mathcal{F})$$

is onto. In particular, there exists a global section \tilde{s}_2 which extends $s_2|_{X \setminus U}$. Arguing as in the proof of Theorem 8, we see that \tilde{s}_2 is actually an extension of s_2 . Thus, $\tilde{s} = s_1 - \tilde{s}_2$ is a section of \mathcal{F} on $U \setminus L$ which extends s . This concludes the proof. \square

Theorem 9 can be slightly improved if X is a manifold. Indeed, in that case, under the same hypothesis for D , we are allowed to assume that \mathcal{F} is defined only in a neighbourhood of \bar{D} .

For the proof we need to recall some classical facts about Function Algebra and envelope of holomorphy (see [6]).

Let X be a complex space and $\mathcal{O}(X)$ be the Fréchet algebra of all holomorphic functions in X . We denote by $\mathcal{S}(X)$ the *spectrum* of $\mathcal{O}(X)$ i.e. the set of all continuous characters $\chi : \mathcal{O}(X) \rightarrow \mathbb{C}$ (or, equivalently, the set of all closed maximal ideals of $\mathcal{O}(X)$) equipped with the weak topology. For every $x \in X$ the point evaluation $f \mapsto \delta_x(f) = f(x)$, $f \in \mathcal{O}(X)$ is a continuous character and $x \mapsto \delta_x$ is a continuous map $i_X : X \rightarrow \mathcal{S}(X)$. Furthermore, for every $f \in \mathcal{O}(X)$ the function $\hat{f} : \mathcal{S}(X) \rightarrow \mathbb{C}$ defined by $\hat{f}(\chi) = \chi(f)$ is continuous and the set $\widehat{\mathcal{O}(X)} = \{\hat{f}\}_{f \in \mathcal{O}(X)}$ is a subalgebra of $C(\mathcal{S}(X))$.

Assume now that X is a Stein space. Then, from Oka-Cartan-Serre theory it follows

- $\alpha)$ i_X is a homeomorphism $X \xrightarrow{\sim} \mathcal{S}(X)$ and there exists a (unique) complex structure on $\mathcal{S}(X)$ such that i_X is a biholomorphism and the dual map $i_X^* : \mathcal{O}(\mathcal{S}(X)) \rightarrow \mathcal{O}(X)$ is an isomorphism; in particular $\widehat{\mathcal{O}(X)} = \mathcal{O}(\mathcal{S}(X))$;

- $\beta)$ for any complex space Y the functors

$$Y \rightarrow \text{Mor}(Y, X), \quad Y \rightarrow \text{Hom}_{\text{cont}}(\mathcal{O}(X), \mathcal{O}(Y))$$

are isomorphic.

A complex space X is said to have an *envelope of holomorphy* if there exists a Stein space \hat{X} with an open immersion $j : X \hookrightarrow \hat{X}$ such that $j^* : \mathcal{O}(\hat{X}) \rightarrow \mathcal{O}(X)$ is an isomorphism of Fréchet algebras. From the properties (α) , (β) it follows that the pair (\hat{X}, j) is uniquely determined (up to isomorphism) by these conditions. Moreover

- $\gamma)$ an envelope of holomorphy of a normal space X is also normal (provided it exists);
- $\delta)$ X has an envelope of holomorphy if and only if $\mathcal{S}(X)$ has a Stein space structure such that $(\mathcal{S}(X), i_X)$ is an envelope of holomorphy of X .

Cartan, Thullen, Oka and Bishop (see [12]) proved that for every Riemann domain over \mathbb{C}^n , $p : \Omega \rightarrow \mathbb{C}^n$, an envelope of holomorphy $\hat{\Omega}$ exists and it is still a domain over \mathbb{C}^n , $\hat{p} : \hat{\Omega} \rightarrow \mathbb{C}^n$. Using the language of the Function Theory this result can be stated as follows (see [6, Theorem 2 and Corollary 1]):

$(\mathcal{S}(\Omega), i_\Omega)$ has a complex manifold structure such that i_Ω is a holomorphic open immersion and $p = \pi \circ i_\Omega$, and $\widehat{\mathcal{O}(X)}$ is the algebra of all holomorphic functions in $\mathcal{S}(\Omega)$. The natural map $\pi : \mathcal{S}(\Omega) \rightarrow \mathcal{S}(\mathbb{C}^n) \simeq \mathbb{C}^n$ is defined by $\chi \mapsto ((\chi(z_1), \dots, \chi(z_n)))$ and is holomorphic of maximal rank. Moreover, $p(\Omega) \subset \pi(\mathcal{S}(\Omega))$.

More generally an envelope of holomorphy exists for domains Ω over a Stein manifold X (see [10]).

For domains in a Stein space X the envelope of holomorphy could not exist even if X is normal, with isolated singularities. The first counterexample is due to Grauert (see [10]).

Theorem 10. *Let D be a bounded domain of a Stein manifold X with a connected smooth boundary and $L \subset bD$ be a Stein compact such that $bD \setminus L$ is connected. Let $\hat{p} : \hat{D} \rightarrow X$ be the envelope of holomorphy of D and \mathcal{F} be a coherent sheaf on a neighbourhood W of $\hat{p}(\hat{D})$ satisfying*

- 1) $\text{depth}(\mathcal{F}_x) \geq 3$ for every $x \in W$;
- 2) $\dim_{\mathbb{C}} T(\mathcal{F}) \leq n - 2$.

Let $U \subset W$ be an open neighborhood of $\overline{D} \setminus L$. Then every section of \mathcal{F} on $U \setminus \overline{D}$ uniquely extends to $D \setminus L$.

Proof. Let \hat{W} be the envelope of holomorphy of W , $\hat{p} : \hat{W} \rightarrow X$ be the canonical projection and $j : W \rightarrow \hat{W}$ be the canonical open embedding of W into \hat{W} . $j^* : \mathcal{O}(\hat{W}) \rightarrow \mathcal{O}(W)$ is an isomorphism. In particular $\hat{p}^* \mathcal{F}$ is a coherent sheaf on \hat{W} with the same depth as \mathcal{F} , which extends $j_* \mathcal{F}$. At this point we argue as in the proof of Theorem 9. \square

3.2. Analytic sets. As for analytic sets, results of removability are obtained arguing as in the proof of Theorem 8 taking into account Theorem 7. Precisely

Theorem 11. *Let X be an n -dimensional manifold, D be a bounded pseudoconvex domain in X with a connected smooth boundary and L be a compact subset of bD . Assume that:*

- 1) $bD \setminus L$ is a connected, non-empty strongly Levi convex hypersurface;
- 2) $D \setminus \hat{L}$ is connected.

Let U be an open neighborhood of $\overline{D} \setminus L$ and Y be a closed, analytic subset of $U \setminus \overline{D}$ such that $\text{depth}(\mathcal{O}_{Y,x}) \geq 3$ for every $x \in U \setminus \overline{D}$. Then Y extends to an analytic subset \tilde{Y} of $(D \setminus \hat{L}) \cup U$.

4. OBSTRUCTIONS TO EXTENSION

The extension theorems proved in the above sections state that, under appropriate conditions, analytic objects like CR -functions, section of coherent sheaves, analytic subsets defined on $bD \setminus L$ ($bD \setminus L$ being connected) extend—uniquely—to $D \setminus \hat{L}$, where \hat{L} is the envelope of L with respect to the algebra $\mathcal{A}(D)$ of holomorphic functions continuous up to the boundary. Natural problems arise about minimality.

In order to state the problem in all generality, given a compact subset L of bD we fix a class \mathcal{C} of analytic objects and we consider the family $\mathcal{L}_{\mathcal{C}}$

of all compact subsets \tilde{L} of \overline{D} , partially ordered by inclusion, satisfying the following properties

- i) $\tilde{L} \cap bD = L$;
- ii) every analytic object of C defined on $bD \setminus L$ extends —uniquely— to $D \setminus \tilde{L}$.

Suppose that $L_C \neq \emptyset$; then exists in L_C some minimal element L_C^0 . One natural problem arises: is L_C^0 unique? In general, due to poldromy phenomena, the answer could be negative. A second observation is that, at least in the cases already considered, if we have unicity then for the minimal compact L_C^0 we have the inclusions

$$L \subset L_C^0 \subset \widehat{L}.$$

The two extremal cases may actually occur. Moreover L_C^0 heavily depends upon the class C . Here are some trivial examples.

- 1) Let $D = \mathbb{B}^n \subset \mathbb{C}^n$ is the unit ball, $L = b\mathbb{B}^n \cap \{\operatorname{Re} z_n \leq 0\}$, $n \geq 3$, and C be the class of holomorphic functions. The minimal compact L_C^0 is $\widehat{L} \supsetneq L$.
- 2) Let $D = \mathbb{B}^n \subset \mathbb{C}^n$, $L = b\mathbb{B}^n \cap \{z_2 = \dots = z_n = 0\} = \mathbb{S}^1 \times \{0\}^{n-1}$, $n \geq 3$, and C be the class of holomorphic functions. The minimal compact L_C^0 is $L \subsetneq \widehat{L}$.
- 3) Let $D = \mathbb{B}^n \subset \mathbb{C}^n$, $L = b\mathbb{B}^n \cap \{z_{n-2} = \dots = z_n = 0\} = \mathbb{S}^1 \times \{0\}^{n-1}$, $n \geq 5$, and C_1 be the class of holomorphic functions, and C_2 be the class of analytic sets of codimension 3. Then the minimal compacts are

$$L_{C_1}^0 = L \subsetneq \widehat{L} = L_{C_2}^0,$$

as shown by the fact that the analytic set

$$\bigcup_{k \in \mathbb{Z}} \left\{ z_{n-2} = z_{n-1} = 0, z_n = \frac{1}{k} \right\}$$

does not extend through \widehat{L} .

5. THE UNBOUNDED CASE

Some of the previous results extend to unbounded domains. The following is of particular interest.

Theorem 12. *Let X be a complex space and D be a strongly pseudoconvex unbounded domain with a connected boundary. Assume that there exists a sequence $\{p_k\}$ of pluriharmonic functions near \overline{D} such that*

- 1) $D_k = \{x \in D : p_k(x) > 0\} \subsetneq D_{k+1} = \{x \in D : p_{k+1}(x) > 0\}$;
- 2) $D_k \Subset X$ and $D = \bigcup_{k \geq 1} D_k$.

Let \mathcal{F} be a coherent sheaf on a neighbourhood U of \overline{D} such that

- 3) $\operatorname{depth}(\mathcal{F}_x) \geq 3$ for every $x \in U$;

$$4) \dim_{\mathbb{C}} T(\mathcal{F}) \leq n - 2.$$

Then every section of \mathcal{F} on $U \setminus \overline{D}$ uniquely extends to a section on U .

Proof. Fix a section σ of \mathcal{F} on $U \setminus \overline{D}$. Consider the domain D_k . Since D is strongly pseudoconvex, using bump lemma we find a Stein neighbourhood $V_k \subset U$ of \overline{D}_k . We may assume that the function p_k is defined on V_k , so $\text{b}D_k \cap \text{b}D$ is a Stein compact L_k , so we are in position to apply Theorem 9 and obtain a unique section $\hat{\sigma}_k$ of \mathcal{F} on $V_k \setminus L_k$ extending σ . Repeating this argument for every k , thanks to uniqueness of extension we get the conclusion. \square

Remark 5.1. If $X = \mathbb{C}^n$, conditions 1), 2) are implied by the following one

- (\star) if \overline{D}^∞ denotes the closure of $D \subset \mathbb{C}^n \subset \mathbb{CP}^n$ in \mathbb{CP}^n , then there exists an algebraic hypersurface V such that $V \cap \overline{D}^\infty = \emptyset$.

Under this condition the extension of analytic sets (with discrete singularities) of dimension at least two holds, see [9].

Acknowledgements. This research was partially supported by MIUR project “Proprietà geometriche delle varietà reali e complesse”.

This paper was partly written during a visit of the authors to Seoul National University in November 2007 and was finished thanks to the hospitality given by Scuola Normale Superiore to the first author. Thus we would like to thank SNU and SNS for the hospitality.

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